The Algebra Project, Feature Talk, and the History of Mathematics

Madeline Muntersbjorn

Abstract Robert P. Moses (1935–2021) was an activist and educator who taught mathematics and studied philosophy. Bob Moses is renowned for his efforts to organize volunteers from around the country who came to Mississippi to help black Americans register to vote. Moses is less widely known as an education reformer, whose philosophy of mathematics shaped the Algebra Project, a curricular approach designed to teach algebra to adolescents. This paper considers what Moses calls "Feature Talk" as opposed to "People Talk." People talk is any natural language. In contrast, feature talk is what results from the mathematization of natural languages when restrictions are placed on how language is used in problem-solving settings to make mathematical relations more tractable. Feature talk is an intermediary between ordinary languages, as commonly spoken, and formal notations, as written by mathematicians and scientists. Feature talk imposes rules that regiment ordinary language, as proposed by Quine. Moses' concept of feature talk is useful for historians of mathematics, especially historians of algebra, who show us how past mathematicians regimented ordinary language in extraordinary ways before the development of algebraic symbolism. Feature talk highlights part of a powerful pedagogical strategy and provides insight into how mathematical practices are cultivated over time.

1 Algebra as a Civil Rights Issue

Bob Moses was a legendary civil rights activist, famous for his efforts organizing the Freedom Summer Project and the Mississippi Freedom Democratic Party (MFDP) in 1964. Never one to put himself in the spotlight, Moses preferred to shine light on his students, whether they were in his Algebra Project classrooms or in the Freedom

M. Muntersbjorn (✉)
University of Toledo, Toledo, OH, USA
e-mail: madeline.muntersbjorn@utoledo.edu

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Schools set up along with voter registration drives in Mississippi. While less famous than the sit-ins, the Freedom Schools not only helped adults learn to read and register to vote but also taught local students everything from history to citizenship, poetry to French (Rothschild 1982). Folks who have heard of Moses, but not the Algebra Project, might presume Moses was a civil rights activist who developed an interest in education later in life. A brief biographical sketch shows Moses was always “all of the above:” mathematics teacher, philosopher of mathematics, and community organizer. Robert P. Moses was born in Harlem and graduated from Stuyvesant High School in 1952, a selective public school in Manhattan, known for excellence in science and math, that continues to offer tuition-free accelerated academics to city residents, as it has since the early twentieth century. Moses earned a bachelor’s degree from Hamilton College in 1956, where he majored in philosophy. Moses earned a master’s degree in philosophy from Harvard in 1957. From 1958 to 1960, Moses taught mathematics at Horace Mann, an elite private school in the Bronx. His studies at Harvard included mathematical logic, but he did not have the certification a job teaching mathematics in the public school system required.

At this time, Moses’ attention was drawn to the Southern United States where he was to undertake the organizational efforts for which he is justly celebrated. Under the tutelage of Ella Baker, Moses became a field secretary for SNCC, the Student Nonviolent Coordinating Committee (pronounced “snick”). His initial forays into Mississippi in the early 1960s, to help blacks register to vote, were met with violent resistance but Moses persisted. In a 1961 letter from jail for disturbing the peace, Moses referred to Mississippi as “the middle of the iceberg” wherein white people in positions of power saw their state as a nation unto itself, not a member of a union wherein the right to vote is federally protected (SNCC Digital Gateway). Under an umbrella organization known as COFO, or the Council of Federated Organizations, Moses coordinated the Freedom Summer Project in 1964 wherein college students from around the country came to Mississippi to enlarge upon the work SNCC had begun. Violent resistance increased even as consensus among diverse groups was built around the idea that the United States Constitution guarantees “One Man, One Vote.”

In the mid-1960s, Moses turned his attention away from SNCC and toward protesting the Vietnam War. In 1966, despite being 31 years old, Moses was drafted, prompting him to leave the country. Moses settled first in Canada and then, from 1969 to 1976, he taught math in Tanzania. Moses returned to the USA in 1976 and resumed his doctoral studies in the philosophy of mathematics at Harvard. Two events coincided in 1982 that led to the Algebra Project. Moses’ eldest daughter went to eighth grade at a public school that did not offer algebra to eighth graders; at the same time, Moses won a MacArthur Fellowship to honor his achievements in the Civil Rights era. This “genius award” gave him the freedom to devote himself to teaching middle school algebra at his daughter’s school (Moses 1995). Moses not only taught algebra at his kids’ school from 1982 to 1987 but also began to promote the idea that access to algebra was the key to mathematical literacy and, ultimately, full citizenship in the twenty-first century. The Algebra Project, Inc. was formally launched in 1990 and has been an active agent of education reform ever since, growing from a handful of students in one school in Cambridge, Massachusetts, to thousands of students in hundreds of schools across the USA.

The argument Moses makes in support of the view that access to algebra is a civil rights issue is based on an analogy. Unregistered voters in the south in the early 1960s were said to be "disinterested" in exercising their right to vote as well as "illiterate" and thus not qualified to vote. When Moses speaks of those days, he recalls the motivation behind the Freedom Schools:

> Education is a basic American value, and it was fundamentally unfair to deny people access to education and to use their lack of education to deny them political access (Moses 2009, p. 377).

Community consensus was built around the idea of one person, one vote as disenfranchised residents of Mississippi learned to read, write, speak, and vote on their behalf. As Moses reminds us:

> … it was not radical in the 60s to do voter registration, but it was radical to do it in the Mississippi Delta among people who were said to be apathetic. Similarly, it is not radical today to teach math using games and enjoyable experiences. What is radical is to teach math to students at the bottom in a way that allows them to rise to the middle and top (Moses et al. 2009, p. 242).

For decades, Moses worked to promote the idea that students who currently score in the bottom quartile on standardized tests deserve an education that prepares them for high school, college, and beyond. The Algebra Project seeks to build consensus around the idea that, "if we can teach algebra to all public-school students then we should." This is a radical proposal, not because it makes mathematics more fun, but because it seeks to make mathematics more accessible to a wider variety of students, especially those who score poorly on state-sponsored assessments.

The objective of the Algebra Project is universal mathematical literacy. Moses suggests we focus on the bottom quartile for whom, evidently, current educational strategies are inadequate. Participants in Algebra Project initiatives are students most at risk of being tracked into non-college preparatory mathematics courses or dropping out of school altogether. What Silva wrote in the project's early days is still true:

> The conviction of the Algebra Project is that *all children can learn algebra*. The Algebra Project challenges the assumptions and practices that deny access to algebra and virtually guarantee failure in higher mathematics and science as a result (Silva et al. 1990, p. 375). [emphasis in original]

The Algebra Project is built on the belief that there are potential scientists, engineers, and financiers in all quartiles, residing in all postal codes. Children who score in the bottom quartile are denied access to the kind of mathematics education they need to succeed in these fields at the college level. The Algebra Project directly challenges the "ability model" that restricts early access to algebra to those who are "gifted" or otherwise "inclined" on the grounds that it is possible to "raise the floor without capping the ceiling," to invoke a useful building metaphor for student achievement (Jetter 1993, p. 28).

In Moses and Cobb, we observe that when folks are asked, "should your child be taught algebra?" the universal reply is affirmative but when asked, "should all children be taught algebra?" answers varied:

> All parents thought *their* child should do algebra, but not all parents thought that *every* child should do algebra (Moses and Cobb 2001, p. 98).

Moses continues:

> The Algebra Project seeks to develop a demand for math literacy in those most affected by its absence – the young people themselves. This approach, which places a high value on the importance of peer culture, is an outgrowth of experience in the civil rights movement of the 1960s, as well as the emergence of Algebra Project graduates into a group with their own perspectives and initiatives (Moses et al. 2009, p. 242).

For example, The Young People's Project (TYPP) grew out of an instantiation of the Algebra Project in Baltimore, Maryland. TYPP offers training to young math literacy workers who teach math to elementary school students at after-school programs and summer camps. Since its inception in 1996, TYPP has spread to ten locations across the USA and has trained thousands of teens and college students to be math literacy workers to assist over 10,000 primary school students with the skills they need to do well in math in secondary school (The Young People's Project 2021).

The Algebra Project is inspired by the political philosophy of American activist Ella Baker (1903–1986) who taught that sustained social change cannot be established from the top down but must be organized from the bottom up through initiatives that make the beneficiaries of social change the architects and enablers of those changes. In her lecture "The Black Woman in the Civil Rights Struggle," Baker famously said:

> In order for us as poor and oppressed people to become part of a society that is meaningful, the system under which we now exist has to be radically changed. This means that we are going to have to learn to think in radical terms. I use the term radical in its original meaning – getting down to and understanding the root cause. It means facing a system that does not lend itself to your needs and devising means by which you change that system (Grant 1998, p. 230).

The lecture from which this quote is taken was delivered at the Institute for the Black World meeting in Atlanta in 1969, the first year Stuyvesant High School in NYC let female students sit their entrance exams. The system that does not lend itself to our needs, that the Algebra Project seeks to change, is a stratified state-by-state patchwork system of US public schools that perpetuate a caste system wherein the best jobs are only career options for the mathematically literate (Moses 2021). Too many poor students are so ill-served by their educations, they can graduate from high school yet struggle to pass remedial mathematics courses at the postsecondary level, significantly, without the prospect of earning college credit.

In a short video for the Children's Defense Fund, Moses (2013) cites three public institutions, one each in California, Maryland, and Florida, where as many as 70%–80% of all first-year students take developmental mathematics for which they earn no college credit because, according to their entrance test scores, they are not ready to take college-level mathematics:

> … these are kids who, in terms of the high schools, are achieving. They are graduating. They have scored whatever the state says they need to score, right, on the state tests. They have scored whatever SAT or ACT says you need to score to get into college, but they get there, and they can't do it. The Algebra Project has been working on a piece of that problem. (See also Chen and Simone 2016).

What do Algebra Project students do? Briefly, Algebra Project students take more math than the minimum required to graduate. While the project originally started with a focus on preparing middle school students to do elementary algebra, the project grew such that to be a part of the project today may entail a more substantial commitment on the part of secondary students. Moses writes,

> With funding from the National Science Foundation, the Algebra Project works within individual classrooms, where we get students to agree to do ninety minutes of math with us every day for four years of high school. They work to catch up their deficits and jump through the country's three hoops: the state hoop, the ACT/SAT hoop, and the university hoop (Moses 2009, p. 379).

Note that the Algebra Project is not about dismantling any hoops. It is not that we must stop assessing students' mathematical skills; rather, we must increase access to mathematical skills. The focus is on getting as many students as possible through as many hoops as possible so that when they graduate from high school, their ambitions will not be curtailed by their lack of mathematical literacy.

The vote and algebra are both tools for progressive social change. Because the Algebra Project is a grassroots project, it involves participation not just from teachers, administrators, and education experts, but from parents and students who must invest time and effort. Sometimes this effort involves catching up on arithmetic skills over the summer; other times it involves attending Saturday academies where parents can get help with their math skills also. Conjoining these interested parties around the shared goal of mathematical literacy makes other positive change possible. As Wahman writes,

> … while algebra and the vote may not appear to be drastic or dramatic enough factors to fundamentally alter the structures of oppression, they are as radical as sit-ins and marches – even more so in that they are likely to achieve long-standing results. … Moses speaks of them as tools that enable people to use rather than be used by the socioeconomic system and to become persons with the power to continually transform their own communities as new needs and problems arise (Wahman 2009, p. 8).

In the same way that SNCC was part of COFO, today the Algebra Project is part of a larger network, We the People, which brings together organizations such as the Young People's Project. We the People was started to promote consensus among diverse groups around the idea that access to a quality mathematics education is a constitutionally guaranteed right insofar as math literacy is necessary for full citizenship in the Information Age, just as reading literacy was necessary in the Industrial Age.

In his 1993 lecture, “Algebra, The New Civil Right,” Moses addressed assembled mathematicians, mathematics educators, and historians of mathematics:

> We are not just talking about professionals and jobs, we really are talking about democracy and citizenship. What is the content of the democracy going to be in this country? These are heavy issues to lay on a group of mathematicians (Moses 1995, p. 58).

On Moses’ view, we need to build consensus around the idea that, if we can teach algebra to all adolescents then we should. Admittedly, this idea is more complex than, one person, one vote. Significantly, communities do not have to buy any specific textbook or purchase any software packages to participate in the Algebra Project. Top-down curricular reform is not the goal. Deciding which course materials to use should involve extensive discussion at the local level. As Moses notes,

> One of the central things we’re saying is that the ongoing struggle for citizenship and equality for minority people is now linked to an issue of math and science literacy. This idea is the background of everything we do in the Algebra Project. … It’s important to make it clear that even the development of some sterling new curriculum – a real breakthrough – would not make us happy if it did not deeply and seriously address the issue of access to literacy for everyone (Moses 1994, p. 107).

In the service of this goal, Moses and colleagues have given considerable thought to how people learn mathematics, especially algebra, and have developed a curricular process that can be implemented in a wide variety of mathematics classrooms.

2 The Algebra Project as a Curricular Process

In an early account of the Algebra Project, Moses (Moses et al. 1989, p. 433) presents a five-step curricular process based on their classroom experiences:

1. Physical event
2. Picture or model of this event
3. Intuitive (idiomatic) language description of this event
4. A description of this event in regimented English
5. Symbolic representation of the event

The process starts from shared experiences of physical events. Moses proposed that many students learn mathematics best by approaching the subject along a “complex, concrete” path as opposed to a “simple, abstract” path:

> That is, if you think of simple as opposed to complex, and abstract as opposed to concrete, we are looking to develop curriculum which finds the right level of complexity in concrete events. We are not building the curriculum around concepts which are simple and abstract. We are trying to build a curriculum around events which are concrete and complex (Moses 1995, p. 65).

The challenge is to manage the complexity by providing rich experiences that can be regimented into the artificial languages of mathematics. Moses is not saying that one could not teach mathematics by leading students along a simple, abstract, and axiomatic path. In Moses (1995, p. 66), he recounts teaching secondary students along this axis at Horace Mann the year after Sputnik launched and set-theoretic approaches to mathematics education became popular. Instead, the Algebra Project seeks to cultivate skills in those students who need to test the mathematics they are learning against their own experiences.

Shared events dispel the air of mystery around the mathematics classrooms where teachers have all the answers, and students are empty vessels to be filled with so many detached facts. Instead, student voices are centered. As Moses said in a podcast,

> So the math class is traditionally a class where the students themselves are disempowered and they sit and listen to an expert, who is the teacher or someone. And so what we set about in the Algebra Project was developing a curricular process to flip that around... (Moses 2020)

For example, to help students learn that mathematicians need to know how to find answers to not only "How many?" questions but also "Which way?" questions, Moses would organize a field trip on public transport (step 1). After the experience, students are encouraged to draw their own pictures of the journey (step 2) and describe these drawings in their own words to one another (step 3). In recounting their journey later, students learn not only to count stops between stations but also to note what direction the stops were in, inbound or outbound. By making their idiomatic descriptions of events more precise and uniform (step 4), students are better able to understand a formal number line, with positive and negative integers (step 5).

On Moses' view, students learn how mathematical practices come into being by actively participating in this process. The group dynamic is critical to Algebra Project pedagogy. Algebra Project secondary school students sign up to be classmates for years at a time for peer support. Together students learn that mathematics is a performative group activity they can do as they practice presenting their thoughts to one another. Because formal symbols come into play so late in the learning process, traditional teachers unfamiliar with Algebra Project pedagogy may not recognize these linguistic activities as mathematics at first. As Payne (2001, p. 249) notes, Moses "believes that it takes a year of exposure before a teacher can make a truly informed decision about whether to buy in." The Algebra Project is not a "quick fix" as lasting mathematical literacy takes considerable patience.

According to Moses, the philosophy behind Algebra Project pedagogy combines two different philosophical insights. The first insight is the cycle of experiential learning, as inspired by Dewey, Piaget, and Lewin.

> Experiential learning theory is grounded in the countless cyclical experiences in which people try something (experience), then think about what they did (reflection), then make improvements (abstract conceptualization), then practice their improvements (application) (Moses et al. 2009, p. 246).

The second philosophical insight is how, precisely, one should think about "abstract conceptualization" as a stage in the experiential cycle. Language-based exercises encourage students to speak about their mathematical experiences, first in their native language, then in formal and regimented language, and finally in mathematical symbols. Step 4 in this process, a description of this event in regimented English, is what came to be known as feature talk in subsequent presentations of the project's curricular process.

When we compare the earlier presentation of the five-step curricular process with versions that came later, we see how the five-step curricular process became more economical. Consider this example of the five-step process:

> When Moses began to work with teachers, he explained how math could be taught as a 'five-step curricular process' in which students should (1) engage in a physical experience, which they could then (2) represent in their own words and pictures, then in structured language, including (3) everyday language ('people talk'), (4) 'feature talk,' and (5) conventional mathematical notation (Davis et al. 2007, p. 71).

Moses notes that feature talk is a conceptual language that no one speaks ordinarily; it is an example of what Quine refers to as regimented languages. Moses and Cobb observe:

> But this is Quine's insight: the symbolic representations in all of mathematics and science are representations of a conceptual language that no people on earth speak as a natural tongue. We came, in time, to call [it] "*feature talk*," an example of what Quine calls "*regimented language*" (Moses and Cobb 2001, p. 201). [emphasis in original]

In people talk, students may say, "Jon is taller than Amy." But in feature talk, students learn to extract content from the comparative relation "taller than" and say, "the height of Jon is greater than the height of Amy," which can more easily be translated into algebraic symbols.

Moses points out that his studies in both philosophy and mathematics gave him insight into how mathematics is cultivated over time. Moses explains:

> I did my work in philosophy of math and I don't think I would have come across thinking like this if I had done my work just in math. And at Harvard in the 1950s, Willard Van Orman Quine [said] ... that elementary arithmetic, elementary set theory, and elementary logic get off the ground by what he called the regimentation of ordinary speech. And that this regimented language was really nobody's spoken language. It was just a form of artificial speak, which was an intermediary between the spoken languages and the actual symbols that are found in the math and science textbooks. ... Now the feature talk, the height of Jon is greater than the height of Amy, can go right over into symbols. All you need is a symbol for height, a symbol for greater than, a symbol for your names, right. ... But the crux of the matter is the move from what Quine would call ordinary language to regimented language (Moses 2020).

According to Ryg, Moses' "Algebraic Pragmatism" is American pragmatism in the tradition of Dewey and Quine, though Ryg is careful to point out,

> The philosophy of Bob Moses is more heavily indebted to Ella Baker than it is to John Dewey. The philosophy of the Algebra Project has been more profoundly influenced by Bob Moses than it has been by Quine (Ryg 2014, p. 291).

Even so, it is worth considering Quine's influence further, especially as we seek to articulate a more precise account of the "crux of the matter," namely, feature talk.

W.V.O. Quine (1908–2000) was influenced by the logical positivism of the Vienna Circle and their careful analyses of the languages of mathematics and science. With respect to his philosophy of mathematics, Quine urged that some, but not necessarily all, mathematical things exist but only in relation to the linguistic structures we rely upon to make them manifest. On Quine's view, how we choose to talk about specific mathematical abstracta will reflect our ontological commitments so we must learn to speak with the utmost clarity if we want to avoid confusion. Further, there is value in cultivating diverse ways of talking about, and thereby committing to, different mathematical ontologies. According to Gregory (2020), while Quine is known as an empirical, naturalist, and pragmatic philosopher, Quine can also be read as a deflationary structuralist about mathematical objects. Gregory observes,

> Regimentation and a critical philosophical eye show us that the domain of objects we are interested in depends somewhat on the questions we are asking (Gregory 2020, p. 111).

Gregory calls our attention to the introduction to section 12 of Quine's *Pursuit of Truth*, "Indifference of Ontology" wherein Quine writes,

> True sentences, observational and theoretical, are the alpha and omega of the scientific. They are related by structure, and objects figure as mere nodes of the structure. (Quine 1992, p. 31 as quoted by Gregory 2020, p. 110)

Mathematical languages are used to transform scientific sentences about observations into theoretical sentences about the logical relations between scientific sentences. These sentences may refer to objects as things in the world, but those specific particulars may be of less interest to us as mathematicians than the more general classes they belong to and the structural relations into which they enter.

Quine's complex answer to the perennial question, "do mathematical entities exist?" has been widely discussed. Less attention has been paid to Quine's answer to the related question, "how does mathematics come into being?" Quine's answer is featured in his essay, "Success and Limits of Mathematization" (Quine 1981, pp. 148–155). On Quine's view, the origins of mathematics are found in the solutions to everyday problems. These solutions often require that ordinary language be regimented or rendered more rigid in its terms and more formulaic in its rules:

> I already urged that set theory, arithmetic, and symbolic logic are all of them products of the straightforward mathematization of ordinary interpreted discourse – mathematization in situ. Set-theoretic laws come of regimenting the ways of reasoning about classes or properties, ways of reasoning that already prevailed more or less tacitly in natural science and ordinary discourse (Quine 1981, p. 150).

Ryg notes that, in Quine's view, mathematization "is continuous with the growth of precision, and it blossoms at last into algorithms and proof procedures" (Ryg 2014, p. 23).

In a posthumously published essay, Moses (2021) argued for a "federal civil rights bill for education, akin to the Civil Rights Act of 1957." The Constitution of the USA must guarantee all citizens an education that allows them to become mathematically literate in an era wherein "algorithms and proof procedures" are ubiquitous. When it comes to mathematics education, federal resources have been invested in assessment and accountability. What must happen so that more resources are invested in access and opportunity? Both We the People and anyone who sees mathematical literacy as necessary for twenty-first-century citizenry face a monumental task. In his lecture at the same conference that Moses spoke at in 1993, historian Victor Katz reminded the audience:

> In fact, throughout most of recorded history, algebra was only taught to the "few," not the "many." So the concept of "algebra for all" is a very recent one that will take a lot of work to make into reality (Katz 1995, p. 17).

3 Feature Talk and the History of Mathematics

Feature talk, as promoted by the Algebra Project, highlights part of a powerful pedagogical strategy and provides insight into how mathematical practices are cultivated. The possibility that the history of algebra yields valuable insights into the teaching of algebra has been explored by Katz (2001) and Katz (2007). For example, Katz avers,

> There are many lessons that history provides for the teaching of algebra, both at the school and university level. First, the subject must have a clear focus, the solving of equations (Katz 2001, p. 358).

One question asked of Algebra Project advocates is, how can an algebra class focus on solving equations and follow the five-step curricular process? Worksheets covered in equations may not be the best way to focus on solving equations. Especially at the introductory level, students struggle with preliminary equations such as $a - b = a + (-b)$. The Algebra Project replies by posing this equation as a why-question when educating communities of parents, teachers, administrators, and others, about Algebra Project pedagogy. Moses observes,

> Posing such a question helps project members learn how people actually think about some of the mathematics that our children must learn, and is simultaneously a way to engage stakeholders in the subject (Moses et al. 2009, p. 239).

By doing this mathematical activity themselves, as a conversational then performative practice, stakeholders experience Algebra Project pedagogy firsthand. Many are surprised to hear professional mathematicians disagree as to why, precisely, $a - b = a + (-b)$. These experiences teach communities how to build consensus around shared mathematical meanings as well as educational goals.

Katz continues the list of lessons to be drawn from the history of algebra for those who teach algebra:

> Second, symbolism should be introduced gradually, as it is needed. Students will eventually discover for themselves the advantages of symbols over words in the context of problem solving, particularly when the verbal procedures become very complicated (Katz 2001, p. 359).

This second lesson from Katz's list underscores the importance of feature talk as an intermediary between ordinary language and formal symbolism. For it is feature talk where verbal statements become complex as precision frequently requires making implicit features explicit, resulting in greater verbosity. This increase in statement length is exactly what good notation abbreviates. It makes no sense to present symbols to students before they are needed to translate long feature talk sentences into more economical equations. The gap from people talk to feature talk must be traversed first before subsequent transition to symbolism will make sense to most students.

The next lesson from Katz is,

> Third, geometrical ideas are always part of algebra and should be used whenever possible, both in the development of algebraic procedures and in developing the notions of analytic geometry (Katz 2001, p. 359).

This lesson reminds us that both algorithms and coordinate systems have geometrical roots. This historical lesson gives credence to the Algebra Project premise that, no matter what concept needs to be taught, one can devise a concrete experience with the right amount of complexity to make that concept accessible to students. Of course, this work of the Algebra Project is never done, and new ideas for classroom activities that exhibit abstract mathematical relations are always in development.

The last lesson from Katz is,

> … algebra teaching needs a constant emphasis on the techniques of problem solving, even if many of the problems are artificial (Katz 2001, p. 359).

This lesson may seem to be where the history of algebra and Algebra Project pedagogy diverge, insofar as the latter prefers "real" problems. However, if the fundamental technique of problem-solving is sharing our thoughts with others while striving to be more rigid in our terminology, then the divergence is superficial. One Algebra Project activity, the Winding Game, teaches students about dividends and remainders using the Chinese zodiac. This game is more artificial, in a sense, than the activity where students learn about ratios by mixing different parts lemon juice, water, and sugar to make lemonade. The goal in both cases is to equip students with mathematical experiences that give meaning to the equations that, after regimentation, represent the relevant magnitudes and relations.

Concrete experiences may need to be part of the pedagogical process:

> To the extent that learning theories are correct about the need for concrete experiences, then the generalizations inherent in drawing and reading maps, building structures, experiencing thrown and falling objects, categorizing characteristics of things, folding paper, and sketching and painting are important components of mathematics education (Katz 2007, p. 199).

These activities may resemble activities outside the math classroom, like cooking, taking public transport, or simply "making do" with what one has on hand. But these experiences may also be "artificial" problems that do not resemble activities outside the math classroom, such as the Winding Game.

Bill Crombie, Director of Professional Learning at the Algebra Project, Inc., described his first reaction to seeing the five-step curricular process in action as gathered stakeholders discussed “making do” stories:

> It felt initially like an English Language Arts exercise. You wrote the stories and compared them, discussed the similarities and differences. And then it was like pulling a rabbit out of a hat; the features that make a make do story are very much the same features of equivalence. It was actually a discussion about mathematics that the middle school teachers were engaged in, about the difference between equality and equivalence (McCallum and Umland 2021).

While it may seem at first like the symbolic equations students learn at the end of the five-step process result from prestidigitation, of course the rabbit was in the hat all along. The mathematical features the teachers want to focus on are built into specific activities from the outset, making the (admittedly daunting) task of regimenting the speech of so many idiosyncratic individuals easier.

Thus far we have considered Algebra Project pedagogy in light of the history of algebra. In conclusion, I consider what lessons historians and philosophers of mathematics can learn from the Algebra Project. The importance of intermediate languages, between ordinary discourse and formal symbolism, has not escaped the notice of astute historians of mathematics. What Moses calls “feature talk,” Chemla (2006) calls “artificial languages” in her paper, “Artificial Languages in the Mathematics of Ancient China.” In her introduction Chemla writes,

> The example presented suggests how artificial languages might result from specific elaborations within, and on the basis of, the usual written or oral language (Chemla 2006, p. 31).

Recall how Moses and Quine portray how mathematics gets started. First, experiences are translated into original sentences in ordinary languages. Then, artificial constraints are imposed upon these sentences so they can function as more precise tools when talking about shared experiences in groups, especially when we seek to abstract the universal and isolate the quantifiable features of these experiences. This regimentation of the ordinary language precedes formal symbolization in the classroom and in the history of mathematics. Chemla concludes by generalizing from her thirteenth-century case study:

> In conclusion, apparently for the sake of mathematical practice, an artificial language, or a sublanguage, was designed within the usual written language. However, its artificial features are not signaled by the use of special symbols. They appear only if we observe the syntax of the expressions thus formed .... Although this conclusion was reached on the basis of a particular example – that of a book written in 13th century China – the phenomenon brought to light seems to me to have a much greater validity: mathematics is always, and has always been, carried out with artificial languages especially designed for carrying out mathematical activity. All these languages must be studied if we are to understand the full implication of such phenomena (Chemla 2006, p. 54).

While the “full implication” of such phenomena may not fit into a single essay, there are some preliminary insights we can draw. The transition from people talk to feature talk is as important as the transition from artificial languages to symbolic formulas. While historians should pay close attention to increased reliance on symbolism in the history of algebra, we also need to better understand those mathematical practices wherein ordinary languages are regimented in extraordinary ways in problem-solving settings. This is not to say that the history of algebra, as a whole, can be cleaved into discrete developmental stages. Rather, Algebra Project pedagogy offers insight into the dynamic dimensions of the growth of mathematics as a group activity. Individual experiences are not translated directly into shared symbols but indirectly via intermediate languages of increasing precision.

Consider the perennial question, is mathematics created or discovered? This question poses a false dilemma insofar as it does not exhaust all possible origin stories for mathematics. If mathematics is purely a social construction, a formal game of signs we make up as we go along, it is hard to explain the success of mathematics in modelling the natural world. Quine's indispensability argument for the reality of at least some mathematical abstracta hinges precisely on this difficulty. If mathematics is eternal, necessary, and out there waiting to be discovered, it is hard to explain how we can know anything about it and, at the same time, explain why we cannot see the whole at once. That is, there are two questions surrounding our epistemological access to timeless and necessary truths facing those who think mathematics is discovered. First, how can we know anything about this eternal necessary realm as finite contingent beings? Second, once we grant that contact is possible somehow, why does it take so long to expand the scope of our mathematical purview? Instead of being able to deduce the whole from a few well-chosen axioms, we must be content to learn mathematics through piecemeal, painstaking, and sometimes disappointing efforts.

Both Moses (1995) and Moses (2012) reference the Frege-Hilbert Controversy surrounding the relations between truth and consistency. In Moses' retelling in Moses (2012), Frege wrote to Hilbert that, "the axioms of geometry are consistent because they are true" while Hilbert wrote back to say, "the axioms of geometry are true because they are consistent." In the decades that followed, "... the battle between Hilbert and Frege really was won by Hilbert." As a result Moses concludes,

> ... what we got in the 20th Century was the elaboration of Hilbert's idea in the teaching of math about consistency, ... so you have certain axioms for school mathematics, and they hang together (Moses 2012).

While Hilbert's formal approach may work for some students, for too many students it does not work. As Moses explains:

> The purpose of the five steps is to avert student frustration in "the game of signs," or the misapprehension that mathematics is the manipulation of a collection of mysterious symbols and signs (Moses et al. 1989, p. 433).

Even if not all mathematics students need to be able to trace symbolic procedures back to lived experiences, all students benefit from experiences that help them better understand how mathematics comes into being as abstract content conveyed by artificial but universal languages anyone can learn to speak, no matter what their native language is.

Moses sides with Frege's realism over Hilbert's formalism for pedagogical reasons. But he also acknowledges that Quine's realism is very different from Frege's. As Moses notes,

> Quine doesn't believe truth as Frege saw it. He has his own version of truth, but he is saying that meaning is attached to language and discourse and so forth (Moses 1995, p. 66).

Perhaps mathematics is neither created nor discovered but cultivated through linguistic processes analogous to cultivation in the agrarian arts. On my view, "wild" mathematical relations are "domesticated" by formal systems of signification that are cultivated in response to practical problems. The regimentation of informal discourse and its translation into more formal feature talk is key to understanding where mathematics comes from. Yes, the regimentation process is artificial. But, like artificial selection, the mathematics that results from our regimentation efforts are not arbitrary. Mathematics comes into being slowly and reflects interactions between our problem-solving ambitions and natural-world constraints.

When thinking about so-called pure mathematics, and whether it is an exception or challenge to this empirical model, it is important to recall the cyclical nature of experiential learning and note there is not one final form that all artificial feature talks must take. Talking about mathematics in precise terms creates new shared experiences that can themselves be discussed, first in ordinary languages, then in more regimented languages, and ultimately in new short-hand symbols and notations that express the new regimented languages more efficiently and explicitly. In this way, what we think of as pure mathematics, detached from lived experiences and real-world applications, may be understood as what results from regimenting how we talk about mathematical experiences to make what is shared and universal in those experiences more salient and manipulable. From this dynamic perspective, all mathematics is applied; what we call pure mathematics is mathematics applied to itself (cf. Steiner 2008, p. 360).

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